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Exact solutions to a schematic nuclear quark model and colorless superconductivity

Henrik Bohr[1] **and João da Providência**[2]

[1] Department of Physics, B 307, Danish Technical University, 2800 Lyngby, Denmark
[2] Departamento de Física, Universidade de Coimbra, 3000 Coimbra, Portugal

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Abstract
Exact solutions are found to the equations of a standard nuclear quark model exemplified by the Bonn model which is defined in terms of an effective pairing force. We show, by symmetry arguments, that, in general, the ground state of this model is not color neutral. In particular, color-neutral states have, in general, higher energy than the ground state. A novel BCS-type formalism, which is able to describe exactly color symmetrical BCS states, is used to show that the model admits, but only as excited states, color-neutral superconductivity. Therefore, such a model, with just a pairing force, is unrealistic as a model for the color-neutral confined phase which prevails at normal nuclear densities. Finally, the paper shows that there exists a color-neutral superconducting phase independently of whether the model is based on the pairing force or a more realistic three-body string force.

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1. Introduction

It has been more or less clear [1], since the creation of QCD as a rigorous theory of strong interaction, that a nuclear structure is based on the more fundamental entities of quark and gluon fields in the framework of QCD field theory. An interesting model (in spite of well-known drawbacks) concerning this viewpoint is the Bonn model [2] that quite successfully describes the nucleus as an MIT bag originally meant for nucleons with a bag pressure ensuring that no free (colored) quarks escape. By introducing a decisive pairing force that suppresses unphysical degeneracies of the quark system, many features of nuclear physics are reasonably well accounted for by this model. Although the model lacks quark confinement, it helps in understanding some symmetry aspects of hadron physics: finite nuclei, nuclear matter or even high-density quark matter. It should be remarked that in 1978 a nuclear physics model which is similar in spirit to the Bonn model was proposed and discussed on the basis of a path integral approach [3]. Even though there are no colored states in QCD, the model contains a colored sector to which the ground state belongs. The existence of a sector which is not color neutral is, of course, an artifact of the theory even with an overall bag pressure. However, the model is interesting because it brings color-neutral states rather close to the ground state. This is quite remarkable since the involved interaction is a two-body force, which is naturally associated with two-body correlations, but not with three-body correlations. On the other hand, it seems reasonable to assume that quarks enter in color-neutral states in the form of color-neutral triplets. For this reason, the model is usually regarded as a model for the formation of color-neutral triplets. Thus, an important ingredient in the Bonn model of Petry *et al* is the referred pairing interaction, inspired by the famous seniority model of nuclear physics [4] that describes superconducting features of a nuclear structure, namely, a characteristic gap in some spectra associated with a pairing force.

This model was originally devised as a model for the formation of triplets (clustering of quarks into nucleons). This being our starting point, we shall derive a series of mathematical results concerning the symmetries of the model and of the states involved. Moreover, we shall arrive at expressions for the ground-state and excited-state energies, and thus for the gap energy. Basically, the conclusion of our study amounts to showing the incompatibility of the model with the assumption of a color-neutral ground state, composed of color-neutral triplets. We find that the model shows a stronger tendency for the formation of Cooper pairs than for the formation of colorless triplets. Moreover, we find that colored BCS states are energetically favored compared with color neutral BCS states, which only exist as excited states. This is in consonance with the existence of a color-superconducting phase in quark matter, recently advocated by many authors [5]. For a recent review, see [6]. A boson condensation mechanism allows for the important color-neutral sector of the model to coexist with the tendency for the formation of Cooper pairs belonging to the $\bar{3}$ representation.

The model is obviously too schematic to be realistic. However, due to the simplicity of the interaction, exact solutions may easily be obtained, so that the model is useful for testing approximation methods which are applied in more realistic models. We find that the BCS wavefunction well describes not only states with color but also those belonging to the color neutral sector. A version of the BCS theory appropriate for describing color neutral states has been presented.

2. The schematic nuclear quark model of Bonn

The Bonn quark model proposed by Petry *et al* [2] contains a schematic interaction of the following form:

$$H_{\text{int}} = G\sum_{i=1}^{3} A_i^* A_i, \tag{1}$$

where

$$A_1^* = \sum_m c_{2m}^* c_{3\tilde{m}}^*, \qquad A_2^* = \sum_m c_{3m}^* c_{1\tilde{m}}^*, \qquad A_3^* = \sum_m c_{1m}^* c_{2\tilde{m}}^*, \qquad \tilde{\tilde{m}} = m \tag{2}$$

and $G < 0$ is the coupling constant. In the following, we shall not treat the full model, but only the interaction part, which amounts to discussing the model in the strong coupling limit. Here, c_{im}^* are quark creation operators and the indices i and m denote, respectively, the color and the remaining single-particle quantum numbers. By $\tilde{m}$, we mean the state obtained from m by time reversal. More specifically, if m stands for the magnetic quantum number, then $\tilde{m} = -m$. Flavor and other quantum numbers may also be included; m then stands for a set of quantum numbers. If m corresponds to momentum $\mathbf{p}$ and flavor f, then $\tilde{m}$ corresponds to momentum

$-\mathbf{p}$ and flavor f. In the two-flavor case, $m = (j_m, \tau)$, where $j_m = -j, -j+1, \ldots, j$ is the magnetic quantum number and $\tau = -\frac{1}{2}, \frac{1}{2}$ is the isotopic spin, then being $\tilde{m} = (-m, -\tau)$, which is a possible choice. This model which is here investigated anew has been further developed by Pittel and others [7], who proposed a mapping of quark degrees of freedom onto collective triplets of constituent quarks. The Hamiltonian consists mainly of a pairing force of the type familiar with BCS models and mimics the 't Hooft potential energy that was developed from a condensate, in the context of non-Abelian gauge theories, like QCD [8]. Clearly, the operators A_i^*, A_i generate a specific algebra. We denote by 2Ω the level degeneracy for a fixed color, that is, the totality of eigenvalues for all quantum numbers beyond color, such as j_z and isospin. In an extended system, and in the spirit of BCS theory, 2Ω stands for the number of states, per color, within the shell on both sides of the Fermi surface where the attractive interaction, responsible for superconductivity, is assumed to act. Then, we find

$$\begin{aligned}
&J_{11} := [A_1^*, A_1] = -2\Omega + \sum_m (c_{2m}^* c_{2m} + c_{3m}^* c_{3m}),\\
&J_{12} := [A_2^*, A_1] = -\sum_m c_{1m}^* c_{2m}, \ldots,\\
&[J_{12}, J_{21}] = \sum_m (c_{1m}^* c_{1m} - c_{2m}^* c_{2m}) = J_{22} - J_{11}, \ldots,\\
&[J_{12}, J_{23}] = \sum_m c_{1m}^* c_{3m} = -J_{13}, \qquad [J_{12}, J_{32}] = 0, \ldots,\\
&[A_1^*, J_{12}] = -\sum_m c_{3m}^* c_{1\tilde{m}}^* = -A_2^*, \ldots .
\end{aligned} \tag{3}$$

In general

$$\begin{aligned}
&[A_i^*, A_j] = J_{ij}, \qquad [J_{ij}, J_{kl}] = J_{kj}\delta_{il} - J_{il}\delta_{kj},\\
&[A_i^*, J_{kl}] = -A_l^*\delta_{ik}, \qquad k \neq l, \qquad [A_i^*, J_{kk}] = -A_i^*(\delta_{ik} + 1),\\
&J_{ij} = -\sum_m c_{im}^* c_{jm} + \delta_{ij}\left(-2\Omega + \sum_{k=1}^{3}\sum_m c_{km}^* c_{km}\right) \qquad i, j, k, l = 1, 2, 3.
\end{aligned} \tag{4}$$

From the number of generators and the commutation relations it is clear that the relevant algebra is $su(4)$. It is natural to define $J_{4i} := A_i^*, J_{i4} := A_i, i = 1, 2, 3$. For $1 \leqslant i, j \leqslant 4$, J_{ij} are the generators of $su(4)$. For $1 \leqslant i, j \leqslant 3$, J_{ij} are the generators of an $su(3)$ sub-algebra which commute with the Hamiltonian H. There are nine generators but only eight are independent, since $\sum_i J_{ii}$ commutes with all of them. Moreover, for $i \neq j$, J_{ij} have the effect of replacing the color j by the color i. If, for $i \neq j$, we have $J_{ij}|\Phi\rangle = 0$, then $|\Phi\rangle$ is color neutral. It should be stressed that, for the same reason, the Hamiltonian H has $su(3)$ symmetry. For curiosity, we may observe that $J_{12}, J_{21}, J_{11}, J_{34}, J_{43}, J_{33}$ generate a $su(2) \times su(2)$ sub-algebra. The eigenvectors of H_{int} are easily determined. Let

$$|\Psi(p_1, p_2, p_3)\rangle = A_1^{*p_1} A_2^{*p_2} A_3^{*p_3}|0\rangle, \qquad p_1 + p_2 + p_3 \leqslant 2\Omega. \tag{5}$$

We find

$$H|\Psi(p_1, p_2, p_3)\rangle = G(2\Omega + 1 - (p_1 + p_2 + p_3))(p_1 + p_2 + p_3)|\Psi(p_1, p_2, p_3)\rangle. \tag{6}$$

Thus, $|\Psi(p_1, p_2, p_3)\rangle$ is an exact eigenvector of H corresponding to the eigenvalue $G(2\Omega+1-(p_1+p_2+p_3))(p_1+p_2+p_3)$. The integers p_1, p_2, p_3 are related to the numbers of quarks in $|\Psi(p_1, p_2, p_3)\rangle$. Hence, p_2+p_3 is the number of quarks of color 1 and $2(p_1+p_2+p_3)$ is the total quark number. The lowest energy $G\Omega(\Omega + 1)$ occurs for $p_1 + p_2 + p_3 = \Omega$. We observe that the state $|\Psi(p_1, p_2, p_3)\rangle$ is not color neutral, even if $p_1 = p_2 = p_3 \neq 0$. Indeed, from (3), it follows that $J_{12}|\Psi(p_1, p_2, p_3)\rangle \neq 0$. Since there are no colored states in QCD, the existence of a sector which is not color neutral must be regarded as an artifact of the model.

In the following sections, we discuss what kind of pairing force we are analyzing, and in section 5 we argue that an appropriate mechanism, such as a three-body string force, is required in order to ensure color neutrality of the ground state. There are two ways of getting a pairing force: one is from outside where 'instantons' or other effects give rise to pairing. Another way is that the pairing phenomena come about because of a weakened string force due to asymptotic freedom.

There is also an important issue concerning the various phases. Usually, one considers the color superconducting phase as being that occurring at low temperature and very high density. However, we have to make clear what symmetries are broken. Another type of color superconductivity seems to be occurring in the confined phase of strings at low temperature and density where the Meissner effect enhances rather than squeezes out the magnetic field. Also electric–magnetic duality changes roles in the color case compared to the QED case.

3. Non-symmetrical representations

If the number of quarks is larger than 2Ω, then it is necessary to consider non-symmetrical representations to reach the ground state. Let

$$|\Phi(\Omega')\rangle = \left(\prod_{m=1}^{\Omega'} c^*_{1m} c^*_{1\tilde{m}}\right)|0\rangle, \tag{7}$$

with $\Omega' \leqslant \Omega$, which means that $2\Omega'$ quarks with color 1 are occupying, in pairs, some of the available single-particle states, but not necessarily all of them. In the product, the notation of the limits, $m = 1$ to Ω', symbolically means that no two m states are considered such that one particle is related to the other by the tilde operation. It is clear that

$$\begin{aligned} A_1|\Phi(\Omega')\rangle &= A_2|\Phi(\Omega')\rangle = A_3|\Phi(\Omega')\rangle \\ &= J_{12}|\Phi(\Omega')\rangle = J_{13}|\Phi(\Omega')\rangle = J_{23}|\Phi(\Omega')\rangle = J_{32}|\Phi(\Omega')\rangle = 0. \end{aligned}$$

Therefore, $|\Phi(\Omega')\rangle$ is appropriate to generate an irreducible representation of $su(4)$, by acting on $|\Phi(\Omega')\rangle$ with $A^*_1, A^*_2, A^*_3, J_{21}, J_{31}$. Let

$$|\Psi(q, \Omega')\rangle = A_1^{*q}|\Phi(\Omega')\rangle. \tag{8}$$

If we restrict our attention to states in which broken pairs are absent, the important states may be reduced either to type (5) or to type (8). The presence of broken pairs is associated with an increase in energy and may be ignored if we wish to focus on the lowest energy states. Then,

$$\begin{aligned} H|\Psi(q, \Omega')\rangle &= Gq(2\Omega + 1 - q)|\Psi(q, \Omega')\rangle \\ &\quad - qA^*_2 A_1^{*(q-1)} J_{21}|\Phi(\Omega')\rangle - qA^*_3 A_1^{*(q-1)} J_{31}|\Phi(\Omega')\rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle\Psi(q, \Omega')|A^*_2 A_2|\Psi(q, \Omega')\rangle &= q^2\langle\Phi(\Omega')|J_{12} A_1^{(q-1)} A_1^{*(q-1)} J_{21}|\Phi(\Omega')\rangle \\ &= q^2(q-1)(2\Omega - q + 1)\langle\Phi(\Omega')|J_{12} A_1^{(q-2)} A_1^{*(q-2)} J_{21}|\Phi(\Omega')\rangle \\ &= q^2\frac{(q-1)!(2\Omega-1)!}{(2\Omega-q)!}\langle\Phi(\Omega')|J_{12}J_{21}|\Phi(\Omega')\rangle \\ &= q^2 2\Omega'\frac{(q-1)!(2\Omega-1)!}{(2\Omega-q)!}, \end{aligned}$$

with a similar result for $\langle\Psi(q,\Omega')|A_3^*A_3|\Psi(q,\Omega')\rangle$, and furthermore

$$\begin{aligned}\langle\Phi(\Omega')|A_1^qA_1^{*q}|\Phi(\Omega')\rangle &= q(2\Omega-q+1)\langle\Phi(\Omega')|A_1^{(q-1)}A_1^{*(q-1)}|\Phi(\Omega')\rangle\\ &= \frac{q!(2\Omega)!}{(2\Omega-q)!}\langle\Phi(\Omega')|\Phi(\Omega')\rangle = \frac{q!(2\Omega)!}{(2\Omega-q)!},\end{aligned}$$

it follows that

$$\frac{\langle\Psi(q,\Omega')|H|\Psi(q,\Omega')\rangle}{\langle\Psi(q,\Omega')|\Psi(q,\Omega')\rangle} = Gq\left(2\Omega+1-q+2\frac{\Omega'}{\Omega}\right). \tag{9}$$

The number of quarks in $|\Psi(q,\Omega')\rangle$ is $2(\Omega'+q)$. For $\Omega'=\Omega$, then $|\Psi(q,\Omega')\rangle$ is an exact eigenvector of H. Indeed, we have $A_2^*|\Phi(\Omega)\rangle=0$, so that

$$A_2^*A_1^{*(q-1)}J_{21}|\Phi(\Omega)\rangle = A_1^{*(q-1)}[A_2^*,J_{21}]|\Phi(\Omega)\rangle = A_1^{*(q-1)}A_1^*|\Phi(\Omega)\rangle = |\Psi(q,\Omega)\rangle.$$

Similarly, $A_3^*A_1^{*(q-1)}J_{21}|\Phi(\Omega)\rangle = |\Psi(q,\Omega)\rangle$. Thus,

$$H|\Psi(q,\Omega)\rangle = Gq(2\Omega+3-q)|\Psi(q,\Omega)\rangle. \tag{10}$$

In this case ($\Omega'=\Omega$), the lowest energy $G\Omega(\Omega+3)$ occurs for $q=\Omega$. Denoting by N the number of quarks, for $N\leqslant 2\Omega$, the ground-state energy is

$$E_0 = GN(4\Omega+2-N)/4, \qquad 0\leqslant N\leqslant 2\Omega. \tag{11}$$

As a function of $N=2(\Omega'+q)$, for fixed Ω', (9) is a parabola. The envelope of these parabolas is the parabolic arc

$$E_0 = G\frac{(N+\Omega+2\Omega^2)^2}{4\Omega(2+\Omega)}, \qquad 2\Omega\leqslant N\leqslant\frac{\Omega(5+4\Omega)}{1+\Omega}, \tag{12}$$

which is, for $2\Omega\leqslant N\leqslant\frac{\Omega(5+4\Omega)}{1+\Omega}$, the approximate ground-state energy, in the sense of the Born approximation. For $\frac{\Omega(5+4\Omega)}{1+\Omega}\leqslant N\leqslant 6\Omega$, the ground-state energy is

$$E_0 = G(N-2\Omega)(6\Omega+6-N)/4, \qquad \frac{\Omega(5+4\Omega)}{1+\Omega}\leqslant N\leqslant 6\Omega. \tag{13}$$

However, no tendency for the formation of colorless triplets really exists. If such a tendency had existed, the ground state would have $su(3)$ symmetry, since H has $su(3)$ symmetry, but this is not the case, except, as we will see, for a narrow interval around $N=3\Omega$. The model lacks an ingredient which automatically pushes colored states to higher energies. The situation is analogous to ferromagnetism, where the ground state explicitly breaks rotational symmetry. We will argue, in the following section, that the formation of Cooper pairs leads to the explicit breakdown of the color symmetry of the model, by most states, including the ground state.

4. Color superconductivity

We propose that this model may be regarded as a model for color superconductivity. We consider the usual color version of the BCS transformation [5],

$$c_{2m} = \alpha d_{2m} + \beta d_{3\tilde{m}}^*, \qquad c_{3m} = \alpha d_{3m} - \beta d_{2\tilde{m}}^*, \tag{14}$$

where α,β are the real parameters such that $\alpha^2+\beta^2=1$. The corresponding BCS vacuum $|\Xi\rangle$ satisfies

$$d_{2m}|\Xi\rangle = d_{3m}|\Xi\rangle = c_{1m}|\Xi\rangle = 0, \tag{15}$$

so that 1 is a spectator color. Obviously, in the above-defined BCS vacuum, the number of quarks of color 1 is zero. The expectation value of the Hamiltonian and of the number of quarks read, respectively,

$$\mathcal{E}_{\rm BCS} = \langle\Xi|H|\Xi\rangle = G((2\Omega)^2\alpha^2\beta^2+2\Omega\beta^4), \qquad \mathcal{N} = \langle\Xi|N|\Xi\rangle = 4\Omega\beta^2. \tag{16}$$

In terms of the average number of quarks, $\mathcal{N}$, the average energy reads

$$\mathcal{E}_{\mathrm{BCS}} = \frac{G}{4}\mathcal{N}\left(4\Omega - \mathcal{N}\left(1 - \frac{1}{2\Omega}\right)\right). \tag{17}$$

Its minimum $G\Omega^2/(1 - 1/(2\Omega))$ occurs for $\mathcal{N} = 2\Omega/(1 - 1/(2\Omega))$. A BCS state containing a non-vanishing number of quarks of color 1 may also be constructed. Define the operators d_{1m} by

$$c_{1m} = \alpha' d_{1m} + \beta' d^*_{1\tilde{m}}, \qquad \alpha'^2 + \beta'^2 = 1, \tag{18}$$

and consider a new BCS vacuum satisfying

$$d_{1m}|\Xi\rangle = d_{2m}|\Xi\rangle = d_{3m}|\Xi\rangle = 0.$$

Then

$$\begin{aligned} \mathcal{E}_{\mathrm{BCS}} &= \langle\Xi|H|\Xi\rangle = G((2\Omega)^2\alpha^2\beta^2 + 2\Omega\beta^4 + 4\Omega\beta^2\beta'^2), \\ \mathcal{N} &= \langle\Xi|N|\Xi\rangle = 4\Omega\beta^2 + 2\Omega\beta'^2 \end{aligned} \tag{19}$$

the number of quarks of color 1 is $\mathcal{N}' = 2\Omega\beta'^2$. The maximum value of $\mathcal{N}'$ is 2Ω. For $\mathcal{N}' = 2\Omega$, the average energy reads, in terms of the average number of quarks,

$$\mathcal{E} = \frac{G}{4}\left(6\Omega + 3 - \mathcal{N}\left(1 - \frac{1}{2\Omega}\right)\right)(\mathcal{N} - 2\Omega). \tag{20}$$

Its minimum $G(\Omega + 1)^2/(1 - 1/(2\Omega))$ occurs for $\mathcal{N} = (4\Omega + 1)/(1 - 1/(2\Omega))$.

We claim that Petry's model describes satisfactorily, in a schematic way, the superconducting phase of quark matter [5]. Our argument relies on the comparison of the results based on the BCS approach with the exact ones. For $\mathcal{N} \leqslant 4\Omega^2/(1 + 2\Omega)$, the BCS ground-state energy is

$$\mathcal{E}_0 = \frac{G}{4}\mathcal{N}\left(4\Omega - \mathcal{N}\left(1 - \frac{1}{2\Omega}\right)\right), \qquad \mathcal{N} \leqslant \frac{4\Omega^2}{1 + 2\Omega}. \tag{21}$$

As a function of $\mathcal{N} = 4\Omega\beta^2 + 2\Omega\beta'^2$, for fixed β', (19) is a parabola. The envelope of this family of parabolas is the parabolic arc

$$\mathcal{E}_0 = G\frac{(\mathcal{N} + 2\Omega^2)^2}{2\Omega(3 + 2\Omega)}, \qquad \frac{4\Omega^2}{1 + 2\Omega} \leqslant \mathcal{N} \leqslant \frac{2\Omega(3 + 4\Omega)}{1 + 2\Omega}, \tag{22}$$

which is the BCS ground-state energy for $\frac{4\Omega^2}{1+2\Omega} \leqslant \mathcal{N} \leqslant \frac{2\Omega(3+4\Omega)}{1+2\Omega}$. For $\frac{2\Omega(3+4\Omega)}{1+2\Omega} \leqslant \mathcal{N} \leqslant 6\Omega$, the BCS ground-state energy is

$$\mathcal{E}_0 = \frac{G}{4}\left(6\Omega + 3 - \mathcal{N}\left(1 - \frac{1}{2\Omega}\right)\right)(\mathcal{N} - 2\Omega), \qquad \frac{2\Omega(3 + 4\Omega)}{1 + 2\Omega} \leqslant \mathcal{N} \leqslant 6\Omega. \tag{23}$$

Comparing (11), (12), (13) with (21), (22), (23), respectively, we see that these results agree with the exact ones, of the order $\frac{1}{2\Omega}$, which supports our claim. In figure 1, we compare, for $\Omega = 10$, the exact ground-state energy versus the fermion number, with the corresponding BCS estimate. The performance of the BCS approximation is remarkable.

Excited states may also be easily described. The energy gap may be defined as the energy difference between ground states corresponding to neighboring irreducible representations. Suppose we replace the vacuum $|0\rangle$ by the state $c^*_{1m'}c^*_{2m'}c^*_{1m'}|0\rangle$. This state behaves as a vacuum on which the pairs $(m', \tilde{m}')$ cannot be produced, so, effectively, the degeneracy becomes $2\,(\Omega - 1)$. For $N = 2\Omega$, the ground-state energy associated with the vacuum $|0\rangle$ was, approximately, $\mathcal{E}_0 = G\big(\big(\Omega + \frac{1}{2}\big)^2$. However, associated with the vacuum $c^*_{1m'}c^*_{2m'}c^*_{1m'}|0\rangle$

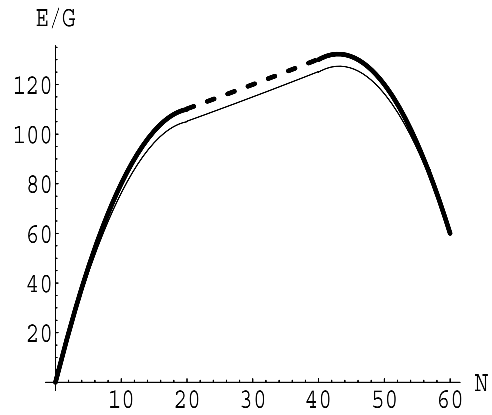

Figure 1. The (absolute) lowest energy of the Bonn model. A comparison of the exact result (thick line) with the BCS estimate (thin line), for $\Omega = 10$. The broken line portion corresponds to a Born approximation estimate.

it becomes $\mathcal{E}_1 = G\left(\Omega - \frac{1}{2}\right)^2$. From

$$\mathcal{E}_1 - \mathcal{E}_0 = G\left(\left(\Omega - \tfrac{1}{2}\right)^2 - \left(\Omega + \tfrac{1}{2}\right)^2\right),$$

the gap $-2G\Omega$ is obtained.

5. Invariant states under $su(3)$

A state which is made up of colorless triplets has the same number of quarks 1, 2, 3. Moreover, it is also invariant under $su(3)$. Let us consider the state

$$|\Phi_W\rangle = \left(\prod_{m=1}^{\Omega'}\prod_{i=1}^{3} c^*_{im}c^*_{i\tilde{m}}\right)|0\rangle, \qquad J_{ij}|\Phi_W\rangle = 0, \quad i \neq j.$$

This is the simplest possible example of a color-neutral state. Contrary to the ground state of H for $N \neq 3\Omega$, this state is invariant under $su(3)$. The number of quarks in $|\Phi_W\rangle$ is $N_W = 6\Omega'$. Since $\langle\Phi_W|H|\Phi_W\rangle = 6\Omega' G$, its energy is $\mathcal{E}_W = 6\Omega' G = N_W G$, which, for $2\Omega < N_W < 4\Omega$ is much higher than the ground-state energy, which is of the order $\left(\Omega + \frac{1}{2}\right)^2 G$ and occurs for a colored state. States made up of triplets have a higher energy (except, as we will see, in a narrow interval around $N = 3\Omega$). The 6-quark combination $\prod_{i=1}^{3} c^*_{im}c^*_{i\tilde{m}}$ is not the only possible one which is color neutral. Other color neutral combinations are possible, such as

$$\prod_{i=1}^{3} c^*_{i_1 m}c^*_{i_1\tilde{m}}\left(c^*_{i_2 m}c^*_{i_3\tilde{m}} + c^*_{i_2\tilde{m}}c^*_{i_3 m}\right)\left(c^*_{i_2 m'}c^*_{i_3\tilde{m}'} + c^*_{i_2\tilde{m}'}c^*_{i_3 m'}\right),$$

where $i_1, i_2, i_3, i = 1, 2, 3$, denote the cyclic permutations of 1, 2, 3. Thus, it is difficult to obtain a useful characterization of the color-neutral sector in the Fermion realization of the $su(4)$ algebra. Nevertheless, a Schwinger-type realization of $su(4)$ in terms of eight bosons has been developed by Yamamura and collaborators [9] which allows a simple and useful characterization of the color-neutral sector for specific quark numbers and degeneracy 2Ω. The model under investigation was originally devised as a model for the formation of triplets (clustering of quarks into nucleons). However, the tendency for the formation of Cooper pairs is much stronger, with BCS states being energetically favored. This tendency may be counterbalanced by an interaction of the type

$$H_{\rm str} = G' \sum_p c_{1p}^* c_{2p}^* c_{3p}^* c_{3p} c_{2p} c_{1p},$$

which mimics the three-body string force introduced in [10], where the existence of such a string force was argued for. We observe that the state $|\Phi_W\rangle$, defined above, is an eigenvector of $H_{\rm str}$. The interaction $H_{\rm str}$ may easily be generalized by replacing the operators c_{im}^*, c_{im} by new Fermi operators d_{im}^*, d_{im} through a canonical transformation. This interaction commutes with the $su(3)$ generators. It is related to the $su(3)$ Casimir operators which do not interfere with the intrinsic properties of the color-neutral sector but are essential for enhancing its role with respect to the colored sectors, by pushing it down in energy.

5.1. Color-neutral superconductivity

Standard two-flavor color superconductivity explicitly breaks color invariance, since pairing is allowed only between quarks with two specific colors and quarks with the third color remaining spectators in the process. The formidable issue of projecting out color-neutral states of the corresponding BCS states is addressed in [11]. As an alternative to the rather involved projection techniques which this approach requires in this subsection, we show that the model also admits color-neutral BCS states, as excited states. The most general color-neutral BCS state reads

$$|\Phi_{cl}\rangle = \exp\left(\kappa \sum_{m=\tilde\Omega}^{\Omega} (c_{1m}^* c_{2\tilde m}^* + c_{2m}^* c_{3\tilde m}^* + c_{3m}^* c_{1\tilde m}^*) + \upsilon \sum_{m>0}^{\Omega} (c_{1m}^* c_{1\tilde m}^* + c_{2m}^* c_{2\tilde m}^* + c_{3m}^* c_{3\tilde m}^*)\right)|0\rangle. \tag{24}$$

In the first sum, the notation from $m = \tilde\Omega$ to Ω symbolically means that no restriction is placed on the single-particle states. In the second sum, the notation for $m > 0$ up to Ω symbolically means that no two m states are considered such that one is related to the other by the tilde operation. We prove that $|\Phi_{\rm cl}\rangle$ is color neutral, in the average, i.e. color quantum numbers balance out to 0. Let us consider the color $su(3)$ generators $S_{ij} = \sum_m \left(c_{im}^* c_{jm} - \frac{1}{3}\delta_{ij}\sum_{k=1}^3 c_{km}^* c_{km}\right)$, which, obviously, are closely related to the previously introduced operators J_{ij}. For $i \neq j$, it is clear that

$$\left[S_{ij}, \left(\kappa \sum_{m=\tilde\Omega}^{\Omega} (c_{1m}^* c_{2\tilde m}^* + c_{2m}^* c_{3\tilde m}^* + c_{3m}^* c_{1\tilde m}^*) + \upsilon \sum_{m>0}^{\Omega} (c_{1m}^* c_{1\tilde m}^* + c_{2m}^* c_{2\tilde m}^* + c_{3m}^* c_{3\tilde m}^*)\right)\right] = 0,$$

which implies

$$S_{ij}|\Phi_{\rm cl}\rangle = 0, \qquad i \neq j.$$

For $i = j$, the analogous commutator does not vanish, which is natural, because the conservation of S_{ii} implies the conservation of the particle number, but this is impossible for a BCS state. However, we still have

$$\langle\Phi_{\rm cl}|S_{ii}|\Phi_{\rm cl}\rangle = 0.$$

In this form, it is explicitly seen that $|\Phi_{\rm cl}\rangle$ is color neutral. However, for analytical development it is more convenient to define $|\Phi_{\rm cl}\rangle$ through the equivalent Bogolubov–Valatin transformation for colorless superconductivity, which involves the three colors on an equal footing and reads

$$\begin{aligned} c_{jm} &= \alpha_{1_j} d_{1m} + \alpha_{2_j} d_{2m} + \alpha_{3_j} d_{3m} + \beta_{1_j} d_{1\tilde m}^* + \beta_{2_j} d_{2\tilde m}^* + \beta_{3_j} d_{3\tilde m}^* \\ c_{j\tilde m} &= \alpha_{1_j} d_{1\tilde m} + \alpha_{2_j} d_{2m} + \alpha_{3_j} d_{3\tilde m} - \beta_{1_j} d_{1m}^* - \beta_{2_j} d_{2m}^* - \beta_{3_j} d_{3m}^*, \\ j &= 1, 2, 3, \end{aligned} \tag{25}$$

where $1_j, 2_j, 3_j$ denote a circular permutation of the indices $1, 2, 3$ that is $(1_1, 2_1, 3_1) = (1, 2, 3), (1_2, 2_2, 3_2) = (2, 3, 1), (1_3, 2_3, 3_3) = (3, 1, 2)$ and the coefficients $\alpha_j, \beta_j, j = 1, 2, 3$, are complex. The transformation is canonical provided

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 = 1$$
$$\alpha_1\alpha_2^* + \alpha_2\alpha_3^* + \alpha_3\alpha_1^* + \beta_1\beta_2^* + \beta_2\beta_3^* + \beta_3\beta_1^* = 0$$
$$\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 - \beta_1\alpha_2 - \beta_2\alpha_3 - \beta_3\alpha_1 = 0.$$

These conditions ensure that the fermion anti-commutation relations

$$\{c_{im}, c_{jm}^*\} = \delta_{i,j}, \qquad \{c_{im}, c_{j\tilde{m}}\} = \{c_{im}^*, c_{j\tilde{m}}^*\} = 0, \tag{26}$$

imply and are implied by the anti-commutation relations

$$\{d_{im}, d_{jm}^*\} = \delta_{i,j}, \qquad \{d_{im}, d_{j\tilde{m}}\} = \{d_{im}^*, d_{j\tilde{m}}^*\} = 0. \tag{27}$$

We easily find, for $j = 1, 2, 3$,

$$\begin{aligned}\tfrac{1}{3}\mathcal{E}_{\text{BCS}} = \langle\Phi_{cl}|A_j^*A_j|\Phi_{cl}\rangle &= 4G\Omega^2|\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1|^2\\ &\quad+ 2G\Omega\big(|\beta_1\beta_2 - \beta_3^2|^2 + |\beta_2\beta_3 - \beta_1^2|^2 + |\beta_3\beta_1 - \beta_2^2|^2\big).\end{aligned} \tag{28}$$

Considering the ansatz

$$\alpha_i = \alpha\,\mathrm{e}^{\mathrm{i}\phi_i}, \qquad \beta_i = \beta\,\mathrm{e}^{\mathrm{i}\phi_i}, \qquad \phi_1 = \phi_2 = -\frac{\pi}{9},$$
$$\phi_3 = \frac{5\pi}{9}, \qquad 3(\alpha^2 + \beta^2) = 1, \qquad \alpha, \beta \geqslant 0,$$

we obtain

$$|\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1|^2 = 3\alpha^2\beta^2, \qquad \left|\beta_{i_1}\beta_{i_2} - \beta_{i_3}^2\right|^2 = 3\beta^2,$$

where i_1, i_2, i_3 denote a permutation of the indices $1, 2, 3$. Finally, we find

$$\begin{aligned}\mathcal{N} &= 18\Omega\beta^2,\\ \mathcal{E}_{\text{BCS}} &= \langle\Phi_{cl}|\sum_{j=1}^{3} A_j^*A_j|\Phi_{cl}\rangle = 36G\Omega^2\alpha^2\beta^2 + 54G\Omega\beta^4\\ &= \frac{G\mathcal{N}}{18\Omega}(12\Omega^2 - (2\Omega - 3)\mathcal{N}).\end{aligned} \tag{29}$$

Comparing with (21), (22) and (23) we conclude that the color-superconducting phase is energetically preferred for any value of $\mathcal{N}$. Using the Schwinger representation of $su(4)$, developed by Yamamura *et al* [9], the lowest energy of the color-neutral sector may be determined. It reads

$$E_{\text{exact}} = G\frac{N}{3}\left(2\Omega + 3 - \frac{1}{3}N\right). \tag{30}$$

In figure 2, the lowest energy of the color-neutral sector, equation (30), is compared with its BCS estimate, equation (29), showing a remarkable performance of the BCS colorless method, which even improves when Ω increases. In the same figure, the prediction of the standard BCS approach, equation (16) is also presented for completeness. This figure shows that the standard BCS approach requires an appropriate extension in order to cover the whole range of fermion numbers, as explained in section 4. The colorless BCS state should be viewed as a colorless condensate of colored BCS pairs.

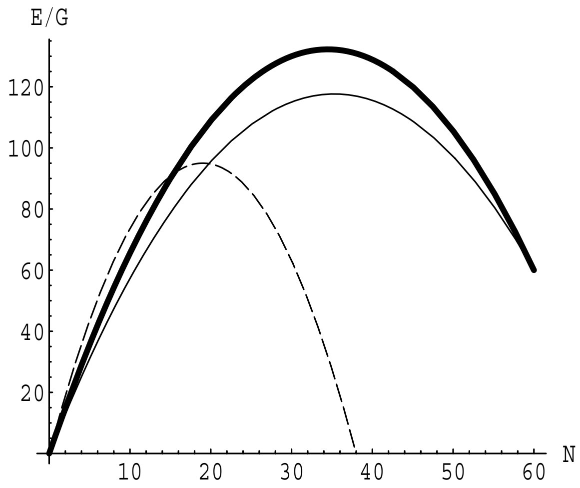

Figure 2. The lowest energy in the color-neutral sector of the Bonn model. A comparison of the exact result (thick line) with the color-neutral BCS estimate, equation (29) (thin line) and conventional BCS, equation (17) (broken line), for $\Omega = 10$.

6. Discussion and conclusions

We have shown that, in the schematic nuclear model which we have investigated, colorless states do not occur at the lowest energies contrary to colored states, so that the model is not compatible with a ground state made up of colorless triplets, except in a narrow interval around $N = 3\Omega$. The model lacks an ingredient which automatically pushes colored states to higher energies. In section 5, we have proposed a more realistic Hamiltonian that provides the foundation for a ground state composed of colorless triplets. It is suggested that such a Hamiltonian should contain a three-body force but it must also involve a chiral σ-field in order to be complete.

The investigated schematic model has been exactly solved. Exact expressions for the ground-state energy and for the gap, which is determined by the degeneracy of the single color level, have also been presented.

It is most interesting to characterize the phases of color superconductivity. It seems that, as mentioned before, a string force is active, but due to asymptotic freedom becomes weak and originates a pairing color force that exists in 9×9 color–flavor combinations. They are in definite color states unless quark–anti-quark pairs are coupled. Thus, we have to explain the color neutrality of the pairing force.

One should make clear what Hamiltonian one uses. Like ordinary superconductivity, there is the Cooper pair interaction from lattice vibrations parallel to the electric Coulomb force. Color is different due to other things than just the change in potential. It to the non-Abelian symmetry that the peculiar behavior of the pairing interaction is due.

In the last few years, several authors have paid attention to the color neutrality phenomenon in superconductivity and QCD [11, 12]. It is worth remarking that in our case we explicitly construct the eigenstates of the non-diagonal generators of color $su(3)$ giving 0 as an eigenvalue. The expectation value of the diagonal generators also vanishes. Indeed, we have presented a color-neutral version of the BCS transformation which leads to exactly color-neutral BCS states. This is in contrast with the approaches followed in [11, 12]. In [11], the authors resort to rather involved projection techniques to extract color-neutral states out of BCS sates which violate color symmetry. It should be pointed out that the correlations described by the present approach need not coincide with those arising within the framework of the projection technique. In [12], color neutrality is defined by the condition that the average or expectation value of some of the eight color operators S_{ij} vanishes, that is, color neutrality is implemented only in the average, even for $i \neq j$.

What is then this colorless state and what does it consist of? It is, as seen from all the vanishing generators, a glueball of vanishing color down to any volume of color strings with gluons adding up to color neutrality, something like a Higgs scalar. A usual understanding of the QCD vacuum is that there are supersymmetric states where electric vortices are squeezed in the same way as the dual corresponding magnetic vortices in QED. Thus, there are string-like color–electric flux lines that actually are neutral which is a very different situation from that in QED.

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Corrigendum

Exact solutions to a schematic nuclear quark model and colorless superconductivity
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Equation (24), expressing the color symmetric BCS state, should be replaced by

$$|\Phi_{cl}\rangle = \exp\sum_{j=0}^{3}\left(K\sum_{0<m\leqslant\Omega'}A^*_{jm} + \tilde{K}\sum_{\Omega'<m\leqslant\Omega}A_{jm}\right)\prod_{j=m}^{3}\prod_{\Omega'<m\leqslant\Omega}c^*_{jm}c^*_{j\overline{m}}|0\rangle,$$

where

$$A^*_{1m} = (c^*_{2m}c^*_{3\overline{m}} + c^*_{2\overline{m}}c^*_{3m}).$$

Equation (25), describing the color symmetric BCS transformation, should be replaced by

$$d_{1m} = \frac{1}{\sqrt{2(1+3K^2)}}(c_{2m} - c_{3m} - K(c^*_{2\overline{m}} + c^*_{3\overline{m}} - 2c^*_{1\overline{m}})),$$

$$d_{2m} = \frac{1}{\sqrt{6(1+3K^2)}}(c_{1m} + c_{3m} - 2c_{2m} + 3K(c^*_{3\overline{m}} - c^*_{1\overline{m}})),$$

$$d_{3m} = \frac{1}{\sqrt{3}}(c_{1m} + c_{2m} + c_{3m}),$$

$$d_{1\overline{m}} = \frac{1}{\sqrt{2(1+3K^2)}}(c_{2\overline{m}} - c_{3\overline{m}} - K(c^*_{2m} + c^*_{3m} - 2c^*_{1m})),$$

$$d_{2\overline{m}} = \frac{1}{\sqrt{6(1+3K^2)}}(c_{1\overline{m}} + c_{3\overline{m}} - 2c_{2\overline{m}} + 3K(c^*_{3m} - c^*_{1m})),$$

$$d_{3\overline{m}} = \frac{1}{\sqrt{3}}(c_{1\overline{m}} + c_{2\overline{m}} + c_{3\overline{m}}),$$

if $0 < m \leqslant \Omega'$, and should be replaced by

$$d_{1m} = \frac{1}{\sqrt{2(1+3\tilde{K}^2)}}(c^*_{2m} - c^*_{3m} + \tilde{K}(c_{2\overline{m}} + c_{3\overline{m}} - 2c_{1\overline{m}})),$$

$$d_{2m} = \frac{1}{\sqrt{6(1+3\tilde{K}^2)}}(c^*_{1m} + c^*_{3m} - 2c_{2m} - 3\tilde{K}(c_{3\overline{m}} - c_{1\overline{m}})),$$

$$d_{3m} = \frac{1}{\sqrt{3}}(c^*_{1m} + c^*_{2m} + c^*_{3m}),$$

$$d_{1\overline{m}} = \frac{1}{\sqrt{2(1+3\tilde{K}^2)}}(c^*_{2\overline{m}} - c^*_{3\overline{m}} + \tilde{K}(c_{2m} + c_{3m} - 2c_{1m})),$$

$$d_{2\overline{m}} = \frac{1}{\sqrt{6(1+3\tilde{K}^2)}}(c^*_{1\overline{m}} + c^*_{3\overline{m}} - 2c^*_{2\overline{m}} - 3\tilde{K}(c_{3m} - c_{1m})),$$

$$d_{3\overline{m}} = \frac{1}{\sqrt{3}}(c^*_{1\overline{m}} + c^*_{2\overline{m}} + c^*_{3\overline{m}}),$$

if $\Omega' < m \leqslant \Omega$.

Equation (29) should be replaced by

$$\mathcal{E}_{\mathrm{BCS}} = \frac{G\mathcal{N}}{9}\left(6\Omega - \mathcal{N} + 1 + \frac{4\mathcal{N}}{3\Omega}\right).$$

For the mathematical development behind these replacements, see a forthcoming note.